JOURNAL OF APPROXIMATION THEORY 45, 310–338 (1985)

The Averaging Method for Asymptotic Evolutions. II. Quantum Open Systems

A. FRIGERIO*

Dublin Institute for Advanced Studies, Dublin 4, Ireland

J. T. LEWIS

Dublin Institute for Advanced Studies, Dublin 4, Ireland

AND

J. V. PULÈ

University College, Dublin 4, Ireland and
Dublin Institute for Advanced Studies, Dublin 4, Ireland

Communicated by Oved Shisha

Received March 14, 1984

The method of averaging is applied to study the reduced evolution of a quantum open system. Successive approximate evolutions are derived, and they are shown to be asymptotic to the exact evolution of the open system, under conditions which are satisfied in the case of a system coupled to a quasi-free reservoir, whose correlation functions are exponentially descreasing in time. © 1985 Academic Press, Inc.

1. INTRODUCTION

In this paper we apply the averaging method described in our previous paper [1] (hereafter referred to as I) to the problem of the reduced evolution of a quantum open system. We have in mind the usual class of models [2–5]: a spatially confined system S, with Hilbert space $\mathscr{H}_S$, is coupled to an infinitely extended reservoir R in a fixed reference state ω_R, which we may represent by a cyclic vector Ω in the GNS space $\mathscr{H}_R$. We let $\mathscr{B}$ be the Banach space of trace class operators on $\mathscr{H}_S \otimes \mathscr{H}_R$; we start with

* Permanent address: Istituto di Scienze Fisiche, Università di Milano, Milano, Italy.

0021-9045/85 $3.00

initial data f_0 in the subspace $\mathscr{B}_0 = \mathscr{T}(\mathscr{H}_S) \otimes |\Omega)(\Omega|$, and we let f_0 evolve in the interaction picture, by putting

$$f^\lambda(t) = V^0_{-t} V^\lambda_t f_0, \qquad f_0 \in \mathscr{B}_0,\ t \geqslant 0, \tag{1.1}$$

where

$$V^\lambda_t b = \exp[-iH_\lambda t]\, b \exp[iH_\lambda t], \qquad b \in \mathscr{B},\ t \in \mathbb{R},$$

and where

$$H_\lambda = H_S \otimes 1_R + 1_S \otimes H_R + \lambda H_I,$$

H_S (resp. H_R) being a self-adjoint operator in $\mathscr{H}_S$ (resp. $\mathscr{H}_R$). H_R is supposed to annihilate Ω; we shall come later on to the conditions that the interaction Hamiltonian H_I must satisfy. Then we project $f^\lambda(t)$ back into $\mathscr{B}_0$ with a projection operator P_0 given by

$$P_0 f^\lambda(t) = \mathrm{tr}_R\{f^\lambda(t)\} \otimes |\Omega)(\Omega|, \tag{1.2}$$

$\mathrm{tr}_R\{\cdots\}$ denoting the partial trace over $\mathscr{H}_R$. We want to find a simple approximate expression for $P_0 f^\lambda(t)$, which is valid for small λ and large t.

Traditionally, this problem has been studied with the use of the generalized master equation (GME), of the form (cf. [2, 3] and references quoted therein)

$$P_0 f^\lambda(t) = f_0 + \lambda^2 \int_0^t U_{-s} \left[\int_0^{t-s} K^\lambda(u)\, du \right] U_s P_0 f^\lambda(s)\, ds, \tag{1.3}$$

where $U_s \rho = \exp[-iH_S s]\, \rho \exp[iH_S s]$, and where the integral kernel $K^\lambda(u)$ has an explicit expression as a power series in λ. It seems very difficult to do something both concrete and rigorous with the GME, which retains memory of all the past history of the system. A great simplification is obtained when the GME is approximated by replacing $\int_0^{t-s} K^\lambda(u)\, du$ with $\int_0^\infty K^\lambda(u)\, du$; the Markovian master equation which then results is expected to give a good approximation when the characteristic relaxation time τ_R of $K^\lambda(u)$ is much smaller than the typical variation time τ_S of $P_0 f^\lambda(t)$, which is of order $1/\lambda^2 \tau_R$. The rigorous theory of the weak coupling limit [2] asserts that, under suitable conditions, there is a time-independent and λ-independent operator G on $\mathscr{B}_0$ such that

$$\lim_{\lambda \to 0} \left\{ \sup_{0 \leqslant \lambda^2 t \leqslant T} \| P_0 f^\lambda(t) - \exp[\lambda^2 G t] f_0 \| \right\} = 0 \tag{1.4}$$

for all f_0 in $\mathscr{B}_0$ and all T in $[0, \infty)$; we have

$$G = \lim_{T \to \infty} \frac{1}{T} \int_0^T U_{-s} \left[\int_0^\infty K^0(u)\, du \right] U_s\, ds. \tag{1.5}$$

In this paper we shall be concerned with the problems of finding a more detailed estimate of the error in (1.4) and of investigating corrections of higher order in λ to $\exp[\lambda^2 Gt] f_0$.

Among the conditions needed to prove (1.4), there is that $P_0([H_I, f_0]) = 0$ for all f_0 in $\mathscr{B}_0$, so that

$$\frac{d}{dt} P_0 f(t)|_{t=0} = 0.$$

Since, on the other hand, we have

$$\frac{d}{dt} \exp[\lambda^2 Gt] f_0|_{t=0} = \lambda^2 G f_0,$$

the best estimate we can hope to obtain in place of (1.4) is

$$\sup_{0 \leqslant \lambda^2 t \leqslant T} \|P_0 f(t) - \exp[\lambda^2 Gt] f_0\| \leqslant \lambda^2 \beta_2(T) \|f_0\| \tag{1.6}$$

for some positive function $\beta_2(\cdot)$, bounded on compacts. The same considerations hold also if one adds some corrections to G, involving higher powers of λ.

It is then clear that, in order to improve the approximation in (1.6), it is necessary to correct the semigroup behaviour, at least for short times, by keeping somehow into account the memory effects which are present in the GME. The method of averaging (I and references quoted therein) provides a convenient alternative approach to the problem, which yields at the same time higher-order terms in G and short time corrections to the semigroup behaviour.

We find it useful to give an idea of the general scheme by presenting a formal derivation (cf. [6] and I). Put

$$f^\lambda(t) = (1 + F^\lambda(t)) \exp[G^\lambda t] f_0,$$

where $F^\lambda(t)$ is an operator from $\mathscr{B}_0$ into $\mathscr{B}$, and G^λ is a time-independent operator on $\mathscr{B}_0$; then one has

$$\frac{d}{dt} f^\lambda(t) = [(1 + F^\lambda(t)) G^\lambda + \dot{F}^\lambda(t)] e^{G^\lambda t} f_0. \tag{1.7}$$

This has to be compared with the differential equation which is equivalent to (1.1),

$$\frac{d}{dt} f^\lambda(t) = -i\lambda[H_I(t), f^\lambda(t)] \equiv \lambda A(t) f^\lambda(t). \tag{1.7'}$$

where $H_I(t) = \exp[iH_0 t]\, H_I \exp[-iH_0 t]$. Identifying (1.7) and (1.7′) gives

$$G^\lambda + \dot{F}^\lambda(t) = \lambda A(t)\, P_0 + \lambda A(t)\, F^\lambda(t) - F^\lambda(t)\, G^\lambda. \tag{1.8}$$

If we try to solve (1.8) by using formal power series

$$F^\lambda(t) = \sum_{m=1}^{\infty} \lambda^m F^{(m)}(t), \qquad G^\lambda = \sum_{m=1}^{\infty} \lambda^m G^{(m)},$$

and equating terms of the same order in λ, we get the hierarchy of equations

$$\begin{aligned} G^{(1)} + \dot{F}^{(1)}(t) &= A(t)\, P_0, \\ G^{(m)} + \dot{F}^{(m)}(t) &= A(t)\, F^{(m-1)}(t) \\ &\quad - \sum_{r=1}^{m-1} F^{(m-r)}(t)\, G^{(r)}, \qquad m = 2, 3, \ldots \end{aligned} \tag{1.9}$$

Then, supposing (1.9) to be satisfied, we have

$$P_0 f^\lambda(t) = (1 + P_0 F^\lambda(t)) \exp[G^\lambda t] f_0. \tag{1.10}$$

The operator $P_0 F^\lambda(t) \equiv M^\lambda(t)$ expresses the deviation of $P_0 f^\lambda(t)$ from exponential behaviour. It represents a "non-Markovian" correction, in that it describes memory effects, although in a more schematic (but more useful) way than the GME (1.3). We use the remaining freedom in the choice of $G^{(m)}$, $F^{(m)}(t)$ satisfying (1.9) to require that this correction remains small for large t, order by order in λ, by asking

$$\lim_{t \to \infty} (1/t)\, P_0 F^{(m)}(t) = 0 \qquad \text{for all} \quad m = 1, 2, \ldots. \tag{1.11}$$

Together with the initial condition $F^{(m)}(0) = 0$ for all m, this requirement determines the solution of the hierarchy (1.9) uniquely. If we define an averaging operation $\mathscr{E}$ on time-dependent operators on $\mathscr{B}$ by

$$\mathscr{E}(B) = \lim_{t \to \infty} \frac{1}{t} \int_0^t P_0 B(s)\, P_0\, ds, \tag{1.12}$$

we can express Eq. (1.11) as $\mathscr{E}(\dot{F}^{(m)}) = 0$, $m = 1, 2, \ldots$, so that $G^{(m)}$ is determined by applying $\mathscr{E}$ to the right-hand side of (1.9), provided, of course, all the integrals and limits involved exist.

We shall consider a class of models for which $G^{(m)}$ and $M^{(m)} = P_0 F^{(m)}$ are non-zero only for m even, giving an expression for $P_0 f^\lambda(t)$ of the form

$$P_0 f^\lambda(t) = (1 + \lambda^2 M^{(2)}(t) + \cdots) \exp[(\lambda^2 G^{(2)} + \lambda^4 G^{(4)} + \cdots)t] f_0. \tag{1.13}$$

The formal power series for G^{λ} and $M^{\lambda}(t)$ are not likely to converge; it may also be the case that $G^{(m)}$, $M^{(m)}(t)$ exist only for m smaller than some fixed n. In order to produce rigorous theorems, we shall proceed as in I, and prove estimates of the form

$$\|P_0 f^{\lambda}(t) - \exp[\lambda^2 G^{(2)} t] f_0\| \leqslant \lambda^2 \beta_2(\lambda^2 t) \|f_0\|, \tag{1.14}$$

$$\begin{aligned} \|P_0 f^{\lambda}(t) - (1 + \lambda^2 M^{(2)}(t)) \exp[(\lambda^2 G^{(2)} + \lambda^4 G^{(4)})t] f_0\| \\ \leqslant \lambda^4 \beta_4(\lambda^2 t) \|f_0\|, \end{aligned} \tag{1.15}$$

where $\beta_n(\cdot)$, $n = 2, 4$, are positive functions, bounded on compacts.

To prove these estimates we shall need some extension of the general theory of I, which we give in Section 2. In Section 2 we discuss also a different version of the averaging method (which is applied, for instance, in [7]), where $P_0 f^{\lambda}(t)$ is regarded as consisting of a slowly varying part exhibiting a semigroup evolution, about which small, rapid oscillations take place. In Section 3, we prove that the conditions stated abstractly and used in Section 2 are indeed satisfied for the usual class of models [2–5], where a spatially confined quantum system (or an N-level system) is coupled to a quasi-free reservoir, consisting of Fermi, Bose, or classical Gaussian fields, with an interaction Hamiltonian H_I which is linear in the reservoir field operators. However, for technical reasons, we need the condition that the two-point correlation functions in the reference state of the reservoir operators appearing in H_I are exponentially decreasing in time, in analogy to I. Some simple illustrative examples, with an explicit calculation of $G^{(2)}$, $M^{(2)}(t)$, and $G^{(4)}$, are given in Section 4. Other applications of the averaging method to quantum open systems may be found in [7, 8]; for those models, the technical condition of exponentially decreasing correlation functions is not satisfied.

We conclude this Introduction with a few remarks. The lowest-order non-vanishing term $G^{(2)}$ in G^{λ} coincides with the operator G of the weak coupling limit theory of [2], hence the estimate (1.14) provides the desired "best bound" (1.6) on the error of the weak coupling approximation. The estimate (1.15) gives the next correction in what looks like an asymptotic expansion of $P_0 f^{\lambda}(t)$ in (even) powers of λ. It seems that higher-order estimates could in principle be obtained under the same assumptions, but the expressions under consideration become complicated very rapidly as the order of the approximation increases.

A more detailed analysis would show that the "asymptotic expansion" of $P_0 f^{\lambda}(t)$ is actually in even powers of λ/α, where α is the decay rate of the exponentially decreasing correlation functions of the reservoir. Notice that $(\lambda/\alpha)^2$ is just the ratio τ_R/τ_S of the two characteristic times in the GME (1.3); the smallness of this ratio is the essential ingredient in all (both nonrigorous and rigorous) discussions of the GME; see [3] for a list of references.

2. General Theory

As in paper I, we let $\mathscr{B}$ be a Banach space, $\mathscr{B}_0$ a closed subspace of $\mathscr{B}$, P_0 a norm one projection of $\mathscr{B}$ onto $\mathscr{B}_0$, $P_1 = 1 - P_0$; let $t \mapsto A(t)$ be a strongly continuous function on $\mathbb{R}^+$ with values in $\mathscr{L}(\mathscr{B})$, and consider the differential equation in $\mathscr{B}$

$$\frac{d}{dt} f^\lambda(t) = \lambda A(t) f^\lambda(t), \qquad t \geqslant 0, \tag{2.1}$$

depending on a parameter $\lambda \in [0, \Lambda]$. Given the initial data f_0 in $\mathscr{B}$, the unique continuous solution on $[0, \infty)$ is given by

$$f^\lambda(t) = U^\lambda(t, 0) f_0, \qquad t \geqslant 0, \tag{2.2}$$

where

$$\begin{aligned} U^\lambda(t, s) &= T \exp\left[\lambda \int_s^t A(u)\, du\right] \\ &= \sum_{n=0}^{\infty} \lambda^n \int_{t \geqslant u_1 \geqslant \cdots \geqslant u_n \geqslant s} \cdots \int A(u_1) \cdots A(u_n)\, du_n \cdots du_1. \end{aligned} \tag{2.3}$$

As explained in the Introduction, we assume $f^\lambda(0) = f_0$ to be in $\mathscr{B}_0$, and we look for an approximate expression for $P_0 f^\lambda(t)$, of the form

$$y_n^\lambda(t) = (1 + M_{n-1}^\lambda(t)) \exp[G_n^\lambda t] f_0, \tag{2.4}$$

where

$$G_n^\lambda = \sum_{m=1}^{n} \lambda^m G^{(m)}, \tag{2.5}$$

$$M_{n-1}^\lambda(t) = \sum_{m=1}^{n-1} \lambda^m M^{(m)}(t) = \sum_{m=1}^{n-1} \lambda^m P_0 F^{(m)}(t), \tag{2.6}$$

and where $F^{(m)}(t)$, $G^{(m)}$ satisfy the hierarchy of equations (1.9), subject to condition (1.11), and with initial condition $F^{(m)}(0) = 0$ for all m. As in I, we shall only consider $n \leqslant 4$. We put

$$A_{ij}(t) = P_i A(t) P_j, \qquad i, j = 0, 1, t \geqslant 0, \tag{2.7}$$

and we make the following assumptions:

(a) $\|U^\lambda(t, s)\| = 1$ for all $t \geqslant s \geqslant 0$;

(b) $P_0 A(t_1) \cdots A(t_{2m+1}) P_0 = 0$ for all $m = 0, 1, \ldots$ and $t_1, \ldots, t_{2m+1} \geqslant 0$;

(c) there are time-dependent operators U_t, $K^{(2)}(t)$, $K^{(4)}(t)$ in $\mathscr{L}(\mathscr{B}_0)$ such that

$$A_{01}(t_1)\, A_{10}(t_2) = -U_{-t_2} K^{(2)}(t_1 - t_2)\, U_{t_2}, \tag{2.8}$$

$$\iint_{t_1 \geq t_2 \geq t_3 \geq t_4} A_{01}(t_1)\, A_{11}(t_2)\, A_{11}(t_3)\, A_{10}(t_4)\, dt_3\, dt_2$$
$$= U_{-t_4} K^{(4)}(t_1 - t_4)\, U_{t_4}, \tag{2.9}$$

where U_t is given by a series

$$U_t = \sum_k e^{i\omega_k t} Q_k, \qquad t \in \mathbb{R}, \tag{2.10}$$

$\{Q_k\}$ being a sequence of operators in $\mathscr{L}(\mathscr{B}_0)$ such that $Q_k Q_l = \delta_{kl} Q_k$ and $\sum_k Q_k = 1$ (strong convergence), and $\{\omega_k\}$ being a sequence of distinct real numbers, with $\inf\{|\omega_k - \omega_l| : k \neq l\} \equiv \delta > 0$.

If U_t commutes with $K^{(2)}(s)$, $K^{(4)}(s)$ for all t in $\mathbb{R}$ and $s \geqslant 0$, we are in the situation described in I. The generalization that we are considering here is made to allow the description of the class of models mentioned in the Introduction for which $\{U_t : t \in \mathbb{R}\}$ is the Hamiltonian evolution of an isolated spatially confined quantum system.

Because of assumption (b), $G^{(m)}$ and $P_0 F^{(m)}(t)$ vanish for m odd, so that the hierarchy of equations (1.9), up to order $n = 4$, becomes

$$\begin{aligned}
\dot{F}^{(1)}(t) &= A(t)\, P_0, \\
G^{(2)} + \dot{F}^{(2)}(t) &= A(t)\, F^{(1)}(t), \\
\dot{F}^{(3)}(t) &= A(t)\, F^{(2)}(t) - F^{(1)}(t)\, G^{(2)}, \\
G^{(4)} + \dot{F}^{(4)}(t) &= A(t)\, F^{(3)}(t) - F^{(2)}(t)\, G^{(2)}.
\end{aligned} \tag{2.11}$$

As in I, we assume the "$(\{c_n\}, \alpha)$-mixing condition":

(d) there are positive constants $\{c_n : n = 0, 1, 2, \ldots\}$, α such that:

(i) the series $\sum_{n=0}^{\infty} c_{2n} z^n$, $\sum_{n=0}^{\infty} c_{2n+1} z^n$ have infinite radius of convergence;

(ii) $$\int \cdots \int_{u \geq v_1 \geq \cdots \geq v_n \geq s} \| A_{01}(u)\, A_{11}(v_1) \cdots A_{11}(v_n)\, R_m^{\lambda}(s) \|\, dv_n \cdots dv_1$$
$$\leqslant c_n (u - s)^{[n/2]} \exp[-\alpha(u - s)], \qquad m = 2, 4,$$

for all $n = 0, 1, \ldots$, where $[n/2]$ is the largest integer not exceeding $n/2$, and where

$$R_m^{\lambda}(s) = A(s)\, F^{(m)}(s) - \sum_{p=1}^{m} \sum_{q=m+1-p}^{m} \lambda^{p+q-m-1} F^{(p)}(s)\, G^{(q)},$$
$$m = 2, 4; \tag{2.12}$$

(iii) $\|K^{(2)}(t)\| \leqslant k_2 e^{-\alpha t}$, $\|K^{(4)}(t)\| \leqslant k_4(t/\alpha) e^{-\alpha t}$, $t \geqslant 0$, for some positive constants k_2 and k_4.

Conditions (a), (b), (c), (d) will be assumed throughout this section, without explicit mention. We put

$$K_{kl}^{(m)}(t) = Q_k K^{(m)}(t)\, Q_l, \qquad m = 2, 4,\ t \geqslant 0. \tag{2.13}$$

THEOREM 1. *The solution of the hierarchy* (2.11), *subject to condition* (1.11) *and with* $F^{(m)}(0) = 0$, *exists and is given by*

$$G^{(1)} = G^{(3)} = P_0 F^{(1)}(t) = P_0 F^{(3)}(t) = 0, \tag{2.14}$$

$$G^{(2)} = -\sum_k \int_0^\infty K_{kk}^{(2)}(t)\, dt, \tag{2.15}$$

$$\begin{aligned} M^{(2)}(t) &= \sum_k \left\{ \int_0^\infty K_{kk}^{(2)}(s) s\, ds + \int_t^\infty (t-s)\, K_{kk}^{(2)}(s)\, ds \right\} \\ &\quad + \sum_{k \neq l} \frac{i}{\omega_k - \omega_l} \int_0^t \left(1 - e^{-i(\omega_k - \omega_l)(t-s)}\right) K_{kl}^{(2)}(s)\, ds, \end{aligned} \tag{2.16}$$

$$G^{(4)} = [G^{(2)}, \bar{M}^{(2)}] + \tilde{G}^{(4)}, \tag{2.17}$$

where $\bar{M}^{(2)}$ *is the time average of* $M^{(2)}(t)$:

$$\bar{M}^{(2)} = \sum_k \int_0^\infty K_{kk}^{(2)}(s) s\, ds + \sum_{k \neq l} \frac{i}{\omega_k - \omega_l} \int_0^\infty K_{kl}^{(2)}(s)\, ds, \tag{2.18}$$

and

$$\begin{aligned} \tilde{G}^{(4)} &= \sum_k \left\{ \int_0^\infty K_{kk}^{(4)}(t)\, dt - \int_0^\infty K_{kk}^{(2)}(s) s\, ds \int_0^\infty K_{kk}^{(2)}(t)\, dt \right\} \\ &\quad + \sum_{k \neq l} \frac{i}{\omega_k - \omega_l} \int_{s=0}^\infty \int_{t=0}^\infty K_{lk}^{(2)}(t)\, e^{i(\omega_k - \omega_l)s} K_{kl}^{(2)}(s)\, ds\, dt. \end{aligned} \tag{2.19}$$

Proof. We have

$$\begin{aligned} G^{(2)} &= \lim_{T \to \infty} \frac{1}{T} \int_{t=0}^T \int_{s=0}^t A_{01}(t)\, A_{10}(s)\, ds\, dt \\ &= -\lim_{T \to \infty} \frac{1}{T} \int_{t=0}^T \int_{u=0}^t \sum_{kl} e^{-i(\omega_k - \omega_l)(t-u)} K_{kl}^{(2)}(u)\, du\, dt. \end{aligned}$$

The terms with $k=l$ give (2.15), and those with $k \neq l$ vanish in the limit as $T \to \infty$. Then we have

$$M^{(2)}(t)=\sum_{k} \int_{s=0}^{t} \int_{u=s}^{\infty} K_{kk}^{(2)}(u)\, du\, ds$$

$$-\sum_{k \neq l} \int_{s=0}^{t} \int_{u=0}^{s} e^{-i(\omega_k-\omega_l)(s-u)} K_{kl}^{(2)}(u)\, du\, ds,$$

which becomes (2.16) with some change of variables.

The hierarchy of equations (2.11) gives for $G^{(4)}$ the expression

$$G^{(4)}=\lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{T} \left\{ \int_{s=0}^{t} U_{-s} K^{(4)}(t-s)\, U_s\, ds \right.$$

$$-\int_{s=0}^{t} U_{-s} K^{(2)}(t-s)\, U_s M^{(2)}(s)\, ds$$

$$\left. +\int_{s=0}^{t} \int_{u=0}^{s} U_{-u} K^{(2)}(t-u)\, U_u G^{(2)}\, du\, ds-M^{(2)}(t)\, G^{(2)} \right\} dt$$

(cf. I). It is clear that the first and the third terms in this expression give the first and the second contributions to (2.19), respectively. The fourth term gives $-\bar{M}^{(2)} G^{(2)}$, and the second gives

$$G^{(2)} \bar{M}^{(2)}-\lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{T} \int_{s=0}^{t} U_{-s} K^{(2)}(t-s)\, U_s\, \Delta M^{(2)}(s)\, ds\, dt,$$

where $\Delta M^{(2)}(s)=M^{(2)}(s)-\bar{M}^{(2)}$ contains a part which vanishes exponentially fast plus an oscillatory part. The first part gives no contribution in the limit as $T \to \infty$, by a change of variables and Lebesgue's dominated convergence theorem. It remains to consider (with a change of variables $t-s=v$)

$$\lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{T} \sum_{j,k,l(k \neq l)} \frac{i e^{-i(\omega_j-\omega_l) t}}{\omega_k-\omega_l}\, dt$$

$$\times \int_{v=0}^{t} e^{i(\omega_j-\omega_k) v} K_{jk}^{(2)}(v)$$

$$\times \int_{u=0}^{t-v} e^{i(\omega_k-\omega_l)(v+u)} K_{kl}^{(2)}(u)\, du\, dv.$$

Only the terms with $j=l$ survive in the limit, giving the last contribution to (2.19).

Now we prove the approximation theorem. Put

$$x_2^\lambda(t) = y_2^\lambda(t) = \exp[\lambda^2 G^{(2)} t] f_0, \tag{2.20}$$

$$x_4^\lambda(t) = \exp[(\lambda^2 G^{(2)} + \lambda^4 G^{(4)}) t] f_0, \tag{2.21}$$

$$y_4^\lambda(t) = (1 + \lambda^2 M^{(2)}(t))\, x_4^\lambda(t), \tag{2.22}$$

where f_0 is in $\mathscr{B}_0$, and where $G^{(2)}$, $M^{(2)}(t)$, $G^{(4)}$ are given by (2.15), (2.16), (2.17), respectively.

THEOREM 2. *There exist positive functions $\beta_n(\cdot)$, $n = 2, 4$, bounded on compact intervals, such that*

$$\|P_0 f^\lambda(t) - y_n^\lambda(t)\| \leqslant \lambda^n \beta_n(\lambda^2 t) \sup_{0 \leqslant s \leqslant t} \|x_n^\lambda(s)\|, \qquad n = 2, 4, \tag{2.23}$$

for all $t \geqslant 0$ and all λ in $[0, \Lambda]$.

Moreover, if the coefficients $\{x_n\}$ in the $(\{c_n\}, \alpha)$-mixing condition satisfy $c_{2n}, c_{2n+1} \leqslant c_0 \kappa^n/n!$, where $\kappa < (\alpha/\lambda)^2$, then the functions β_n can be put in the form $\beta_n(t) = a_n + b_n t$, a_n, $b_n > 0$, $n = 2, 4$; and there is a constant $c > 0$ such that

$$\|P_0 f^\lambda(t) - x_4^\lambda(t)\| \leqslant \lambda^2 (c + b_4 \lambda^4 t) \sup_{0 \leqslant s \leqslant t} \|x_4^\lambda(s)\| \tag{2.24}$$

for all $t \geqslant 0$ and all λ in $[0, \Lambda]$.

Proof. When U_t commutes with $K^{(n)}(s)$, $n = 2$, 4, for all t in $\mathbb{R}$ and $s \geqslant 0$, this follows from Theorems 1 and 2 of I. The additional complication of oscillatory terms if U_t does not commute with $K^{(n)}(s)$ is only reflected in the form of $G^{(2)}$, $M^{(2)}(t)$, $G^{(4)}$; its effects are given in Theorem 1 above. The norm estimates on the errors are not affected by the presence of the isometries U_t, and they are derived in the same way as in I.

Comparing Theorems 1 and 2 here with the corresponding results of I, we see that now the problem has three characteristic times: the inverse coupling constant $1/\lambda$, the decay time $1/\alpha$ of the reservoir correlation functions, and the characteristic time $1/\delta$ for oscillations in the system, defined by $\delta = \min\{|\omega_k - \omega_l| : k \neq l\}$. Norm bounds on $G^{(2)}$, $G^{(4)}$, $M^{(2)}(t)$ are of the form

$$\begin{gathered}
\lambda^2 \|G^{(2)}\|\, t \leqslant k_2(\lambda/\alpha)\, \lambda t, \\
\lambda^4 \|G^{(4)}\|\, t,\ \lambda^4 \|\tilde{G}^{(4)}\|\, t \leqslant 2[(k_2^2 + k_4)(\lambda/\alpha)^3 + k_2^2 \lambda^3/\alpha^2 \delta]\, \lambda t, \\
\lambda^2 \|\bar{M}^{(2)}\| \leqslant k_2(\lambda/\alpha)(\lambda/\alpha + \lambda/\delta), \\
\lambda^2 \|M^{(2)}(t) - \bar{M}^{(2)}\| \leqslant k_2[(\lambda/\alpha + \lambda/\delta)\, e^{-\alpha t} + \lambda^2/\alpha\delta].
\end{gathered} \tag{2.25}$$

Hence the coupling constant λ has to be small in comparison to both α and δ, in order for the estimates of Theorem 2 to be of interest.

As in paper I, the well-known results of the weak coupling limit theory [2] can be recovered from our estimate (2.23) for $n=2$; we obtain

$$\lim_{\lambda \to 0} \|P_0 f^{\lambda}(\tau/\lambda^2) - \exp[G^{(2)}\tau] f_0\| = 0 \tag{2.26}$$

uniformly on all compact intervals $0 \leqslant \tau \leqslant \tau_1$. As a consequence, $\exp[G^{(2)}\tau]$ is a contraction for all positive τ, being a limit of contractions; and the estimate (1.14) of the Introduction follows. Similarly, (1.15) would follow from (2.23) for $n=4$, if we know that $\exp[(G^{(2)} + \lambda^2 G^{(4)})\tau]$ is a contraction for all positive τ; this will be the case in the applications we shall consider; in the general case, an estimate of the form (1.15) follows if $\beta_4(\lambda^2 t)$ is replaced by $\beta_4(\lambda^2 t) \exp[\lambda^2 t \, \|G^{(2)} + \Lambda^2 G^{(4)}\|]$.

It is sometimes useful to regard $P_0 f^{\lambda}(t)$ as given by a slowly varying part $\tilde{x}^{\lambda}(t)$, exhibiting a semigroup evolution, about which small, rapid oscillations take place; it is then argued that only the rate of change of $\tilde{x}^{\lambda}(t)$ is accessible to measurement (for instance, this is the point of view taken in [7]). In the present framework, this picture can be substantiated as follows: we replace $P_0 f^{\lambda}(t)$ with its approximate expression $y_4^{\lambda}(t)$, we split $M^{(2)}(t)$ as its time average $\bar{M}^{(2)}$ plus an additional term $\Delta M^{(2)}(t)$, and we commute $(1 + \lambda^2 \bar{M}^{(2)})$ with $\exp[(\lambda^2 G^{(2)} + \lambda^4 G^{(4)})t]$, taking into account the (in general non-vanishing) commutator between $G^{(2)}$ and $\bar{M}^{(2)}$. The slowly varying part of $y_4^{\lambda}(t)$ is then obtained by neglecting the term containing the rapidly varying, zero-average expression $\Delta M^{(2)}(t)$.

THEOREM 3. *Let λ^2 be strictly smaller than $\|\bar{M}^{(2)}\|$. Then there is a positive function $\tilde{\beta}_4(\cdot)$, bounded on compacts, such that*

$$\|P_0 f^{\lambda}(t) - (1 + \lambda^2 \, \Delta M^{(2)}(t)) \, \tilde{x}_4^{\lambda}(t)\| \leqslant \lambda^4 \beta_4(\lambda^2 t) \, \|f_0\|, \tag{2.27}$$

where $\Delta M^{(2)}(t) = M^{(2)}(t) - \bar{M}^{(2)}$, and where

$$\tilde{x}_4^{\lambda}(t) = \exp[(\lambda^2 G^{(2)} + \lambda^4 G^{(4)})t](1 + \lambda^2 \bar{M}^{(2)}) f_0, \qquad t \geqslant 0. \tag{2.28}$$

Proof. If $\lambda^2 \|\bar{M}^{(2)}\| < 1$, then $(1 + \lambda^2 \bar{M}^{(2)})^{-1}$ exists and is given by a convergent power series expansion. The quantity $y_4(t)$ may be rewritten as

$$\begin{aligned} y_4^{\lambda}(t) &= (1 + \lambda^2 M^{(2)}(t))(1 + \lambda^2 \bar{M}^{(2)})^{-1} \\ &\quad \times \exp[\lambda^2 t (1 + \lambda^2 \bar{M}^{(2)})(G^{(2)} + \lambda^2 G^{(4)})(1 + \lambda^2 \bar{M}^2)^{-1}](1 + \lambda^2 \bar{M}^{(2)}) f_0. \end{aligned}$$

Now we have

$$\|(1 + \lambda^2 M^{(2)}(t))(1 + \lambda^2 \bar{M}^{(2)})^{-1} - (1 + \lambda^2 \, \Delta M^{(2)}(t))\| = O(\lambda^4),$$

uniformly in t, and

$$\begin{aligned}\|(1+\lambda^2\bar{M}^{(2)})(G^{(2)}+\lambda^2 G^{(4)})(1+\lambda^2\bar{M}^{(2)})^{-1}\\ -\{G^{(2)}+\lambda^2(G^{(4)}+[\bar{M}^{(2)},G^{(2)}])\}\|=O(\lambda^4),\end{aligned}$$

so that, using (2.17) and the usual kind of estimates for the approximation of semigroups (cf. [11, Chap. IX, Sect. 2]), we find

$$\|y_4^\lambda(t)-(1+\lambda^2\,\Delta M^{(2)}(t))\,\tilde{x}_4^\lambda(t)\|\leqslant\lambda^4(\text{const})\exp[\lambda^2 t(\text{const})],$$

as required. The exponential bound could be replaced by a linear bound, of the form $\leqslant\lambda^4(a_4+\lambda^2 tb_4)$, if the operators $G^{(2)}+\lambda^2 G^{(4)}$, $G^{(2)}+\lambda^2\tilde{G}^{(4)}$ were generators of semigroups of contractions, and this will be the case in our applications.

The expression $\tilde{x}_4^\lambda(t)$ given by (2.28) is interpreted as the slowly varying part of $P_0 f^\lambda(t)$, up to fourth order in λ. It obeys a semigroup evolution law, with generator $\lambda^2 G^{(2)}+\lambda^4 G^{(4)}$, and with a shifted initial condition $(1+\lambda^2\bar{M}^{(2)})f_0$. Notice that the difference between $P_0 f^\lambda(t)$ and $\tilde{x}_4^\lambda(t)$ contains a term of order λ^2, which is neglected in various applications on the grounds that it is rapidly oscillating (cf. [7]).

In [7] the expression of $\lambda^2 G^{(2)}+\lambda^4 G^{(4)}$ is derived assuming the system and the reservoir to be mutually uncorrelated in the remote past ($F^{(m)}(t)\to 0$ as $t\to-\infty$). Here we show that both procedures give the same result.

THEOREM 4. *The expressions* (2.15), (2.17) *of* $G^{(2)}$, $G^{(4)}$ *may be obtained by solving the hierarchy of equations* (2.11) *subject to the conditions*

$$\mathscr{E}(G^{(m)})=G^{(m)},\qquad \mathscr{E}(\dot{F}^{(m)})=0,\qquad F^{(m)}(t)\to 0\qquad \text{as } t\to-\infty, \tag{2.29}$$

provided one interprets $\lim_{t\to\pm\infty} e^{i\omega t}$ *in the sense of distributions* ($=0$ *for real* $\omega\neq 0$).

Proof. We repeat the computations of Theorem 1, with the new condition on $F^{(m)}(t)$. Then $F^{(m)}(t)=\int_{-\infty}^t \dot{F}^{(m)}(s)\,ds$, where $\int_{-\infty}^t e^{i\omega s}\,ds$ is interpreted as $e^{i\omega t}/i\omega$ for all real $\omega\neq 0$. So we have

$$G^{(2)}=\lim_{T\to\infty}\frac{1}{T}\int_{t=0}^T\int_{s=-\infty}^t A_{01}(t)\,A_{10}(s)\,ds\,dt.$$

We put $u=t-s$; then u goes from 0 to ∞, and we get

$$G^{(2)}=-\lim_{T\to\infty}\frac{1}{T}\int_{t=0}^T\int_{u=0}^\infty\sum_{kl} e^{-i(\omega_k-\omega_l)(t-u)}\,K_{kl}^{(2)}(u)\,du\,dt. \tag{2.30}$$

The terms with $k=l$ give (2.15) again, and those with $k \neq l$ vanish in the limit as $T \to \infty$. We have also

$$M^{(2)}(t) = \int_{s=-\infty}^{t} \left[\int_{u=-\infty}^{s} A_{01}(s)\, A_{10}(u)\, du - G^{(2)} \right] ds$$

$$= -\int_{s=-\infty}^{t} \int_{v=0}^{\infty} \sum_{k \neq l} e^{-i(\omega_k - \omega_l)(s-v)} K_{kl}^{(2)}(v)\, dv\, ds$$

$$= -\sum_{k \neq l} \frac{i}{\omega_k - \omega_l} e^{-i(\omega_k - \omega_l)t} \int_0^{\infty} e^{+i(\omega_k - \omega_l)v} K_{kl}^{(2)}(v)\, dv,$$

which is the long time behaviour of the oscillating part of (2.17). Then $\bar{M}^{(2)} = 0$, and

$$G^{(4)} = \lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{T} \left\{ \int_{s=-\infty}^{t} U_{-s} K^{(4)}(t-s)\, U_s\, ds \right.$$

$$+ \int_{s=-\infty}^{t} U_{-s} K^{(2)}(t-s)\, U_s \sum_{k \neq l} \frac{i}{\omega_k - \omega_l} \int_0^{\infty} e^{-i(\omega_k - \omega_l)(s-v)} K_{kl}^{(2)}(v)\, dv\, ds$$

$$\left. + \int_{s=-\infty}^{t} \int_{u=-\infty}^{s} U_{-u} K^{(2)}(t-u)\, U_u G^{(2)}\, du\, ds \right\} dt.$$

We compute separately the three functions of t to be averaged. The first is

$$\sum_{kl} \int_{u=0}^{\infty} e^{-i(\omega_k - \omega_l)(t-u)} K_{kl}^{(4)}(u)\, du,$$

like in (2.30); the second is

$$\sum_{j,k,l(k \neq l)} \int_{u=0}^{\infty} \int_{v=0}^{\infty} e^{-i(\omega_j - \omega_l)(t-u)} K_{jk}^{(2)}(u) \frac{i e^{i(\omega_k - \omega_l)v}}{\omega_k - \omega_l} K_{kl}^{(2)}(v)\, dv\, du,$$

and the third is

$$\int_{u=-\infty}^{t} U_{-u} K^{(2)}(t-u)\, U_u (t-u)\, G^{(2)}\, du$$

$$= \sum_{kl} \int_{s=0}^{\infty} e^{-i(\omega_k - \omega_l)(t-s)} K_{kl}^{(2)}(s)\, s G^{(2)}\, ds.$$

The only terms surviving the average operation are those with $k=l$ in the first and in the third functions, and those with $j=l$ in the second function. The result is exactly (2.19).

3. Estimates

As is the case for the weak coupling limit theory [2–5], with the only exception of [10], we are able to check the required mixing conditions for the reservoir only when the latter is quasi-free. We shall only consider a coupling H_I which is linear in the reservoir field operators; we shall discuss first the fermion case, where H_I is bounded.

Let $\mathcal{A}_R$ be the Clifford algebra over a complex Hilbert space $\mathscr{V}$, generated by bounded self-adjoint field operators $\{\phi(v): v \in \mathscr{V}\}$ satisfying the anticommutation relations

$$\phi(v)\,\phi(v') + \phi(v')\,\phi(v) = 2\,\mathrm{Re}(v, v')1 \qquad \text{for all } v, v' \text{ in } \mathscr{V}. \tag{3.1}$$

Let ω_R be a quasi-free state on $\mathcal{A}_R$, with correlation functions

$$\omega_R(\phi(v_1)\cdots\phi(v_{2m+1})) = 0$$
$$\text{for all } m = 0, 1,\dots \text{ and for all } v_1,\dots, v_{2m+1} \text{ in } \mathscr{V}; \tag{3.2}$$

$$\omega_R(\phi(v_1)\cdots\phi(v_{2m}))$$
$$= \sum_{p \in \mathscr{P}_m} \operatorname{sgn} p \prod_{q=1}^{m} \omega_R(\phi(v_{p(2q-1)})\,\phi(v_{p(2q)}))$$
$$\text{for all } m = 1, 2,\dots \text{ and for all } v_1,\dots, v_{2m} \text{ in } \mathscr{V}, \tag{3.3}$$

where $\mathscr{P}_m$ is the set of those permutations p of $\{1,\dots, 2m\}$ such that $p(2q-1) < p(2q)$ and $p(2q-1) < p(2q+1)$ for all q, and $\operatorname{sgn} p$ is the parity of the permutation p.

Let also $\{T_t: t \in \mathbb{R}\}$ be a strongly continuous group of unitaries on $\mathscr{V}$, such that

$$\omega_R(\phi(T_t v)\,\phi(T_t v'))$$
$$= \omega_R(\phi(v)\,\phi(v')) \qquad \text{for all } v, v' \text{ in } \mathscr{V} \text{ and all } t \text{ in } \mathbb{R}. \tag{3.4}$$

Then let $(\mathscr{H}_R, \pi_R, \Omega_R)$ be the GNS triple associated to the state ω_R on $\mathcal{A}_R$. We identify $\mathcal{A}_R$ with its image under π_R, and we write simply Ω for Ω_R. There is a self-adjoint operator H_R in $\mathscr{H}_R$ such that $H_R\Omega = 0$ and

$$\exp[iH_R t]\,\phi(v)\exp[-iH_R t] = \phi(T_t v) \qquad \text{for all } v \text{ in } \mathscr{V},\ t \text{ in } \mathbb{R}. \tag{3.5}$$

Let $\mathscr{H}_S$ be a separable Hilbert space. We interpret $\mathcal{A}_S = \mathscr{L}(\mathscr{H}_S)$ as the algebra of observables of the system S, and $\mathcal{A}_R$ as the algebra of observables of the reservoir R.

Let H_S be a self-adjoint operator in $\mathscr{H}_S$, whose spectrum is pure point and has no finite accumulation points. Let $\{X_j: j = 1,\dots, r\}$ and

$\{v_j: j = 1, ..., r\}$ be finite collections of bounded self-adjoint operators on $\mathscr{H}_S$ and of vectors in $\mathscr{V}$, respectively, and put

$$H_I = \sum_{j=1}^{r} X_j \otimes \phi(v_j); \tag{3.6}$$

then H_I is a bounded self-adjoint operator on $\mathscr{H}_S \otimes \mathscr{H}_R$.

Then we may proceed as stated in the Introduction. Equation (1.7′) holds with

$$A(t)f = -i \sum_{j=1}^{r} [e^{iH_S t} X_j e^{-iH_S t} \otimes \phi(T_t v_j), f]; \qquad f \in \mathscr{B}. \tag{3.7}$$

Put

$$X_j(t) = e^{iH_S t} X_j e^{-iH_S t}, \qquad j = 1, ..., r, \; t \in \mathbb{R}, \tag{3.8}$$

$$\begin{aligned} h_{ij}(t) &= \langle \phi(v_i)\, \phi(T_t v_j) \rangle \equiv \omega_R(\phi(v_i)\, \phi(T_t v_j)) \\ &= \overline{h_{ji}(-t)}, \qquad i, j = 1, ..., r, \; t \in \mathbb{R}. \end{aligned} \tag{3.9}$$

We identify $\mathscr{B}_0 = \mathscr{T}(\mathscr{H}_S) \otimes |\Omega)(\Omega|$ with $\mathscr{T}(\mathscr{H}_S)$. It is clear that conditions (a), (b), (c) of Section 2 hold, with

$$U_t \rho = e^{-iH_S t} \rho\, e^{iH_S t}, \qquad \rho \in \mathscr{T}(\mathscr{H}_S), \; t \in \mathbb{R}, \tag{3.10}$$

$$K^{(2)}(t)\rho = -\sum_{i,j=1}^{r} [X_i(t), \rho X_j]\, h_{ji}(t) + \text{h.c.}, \tag{3.11}$$

$$\begin{aligned} K^{(4)}(t) = \sum_{i,j,k,l=1}^{r} \int_{s=0}^{t} \int_{u=0}^{s} & \{[X_i(t), X_j(s)\, X_k(u)\, X_l \rho] \\ & \times (h_{il}(-t)\, h_{jk}(u-s) - h_{ik}(u-t)\, h_{jl}(-s)) \\ & - [X_i(t), X_j(s)\, X_k(u)\, \rho X_l](h_{li}(t)\, h_{jk}(u-s) - h_{ik}(u-t)\, h_{lj}(s)) \\ & - [X_i(t), X_j(s)\, X_l \rho X_k(u)](h_{ki}(t-u)\, h_{jl}(-s) - h_{il}(-t)\, h_{kj}(s-u)) \\ & - [X_i(t), X_k(u)\, X_l\, \rho X_j(s)] \\ & \times (h_{ik}(u-t)\, h_{jl}(-s) - h_{il}(-t)\, h_{jk}(u-s)) + \text{h.c.}\}\, du\, ds, \end{aligned} \tag{3.12}$$

where ρ is a self-adjoint element of $\mathscr{T}(\mathscr{H}_S)$, $t \geq 0$, and where h.c. denotes hermitian conjugate.

The above considerations remain true, with minor modifications, also in the case of an unbounded Hamiltonian H_I of the form (3.6), for which the operators $\phi(v_j)$ are classical Gaussian variables, or Bose field operators satisfying the commutation relations

$$\phi(v)\, \phi(v') - \phi(v')\, \phi(v) = 2i \,\mathrm{Im}(v, v')\mathbf{1} \qquad \text{for all } v, v' \text{ in } \mathscr{V}. \tag{3.13}$$

In any case, the $\phi(v)$ are densely defined self-adjoint operators in a Hilbert space $\mathscr{H}_R$, with a cyclic vector Ω which is in the domain of all monomials $\phi(v_1)\cdots\phi(v_n)$; putting $\omega_R(\cdots)=(\Omega,\ldots\,\Omega)$, (3.2) still holds, and (3.3) holds with (sgn p) omitted. We also assume (3.4) to be true, then (3.5) holds.

Then all the theory of I and of Section 2 above remains valid (cf. I, Remark A), the reason being essentially that the expansion

$$P_0 f(t)=f_0+\sum_{n=1}^{\infty}\int\cdots\int_{t\geqslant t_1\geqslant\cdots\geqslant t_n\geqslant 0} P_0 A(t_1)\cdots A(t_n)\, f_0\, dt_n\cdots dt_1$$

is well defined and convergent for all f_0 in $\mathscr{B}_0$, $t\geqslant 0$. The operators U_t and $K^{(2)}(t)$ are still given by Eqs. (3.10) and (3.11), respectively, and $K^{(4)}(t)$ has an expression which is similar to (3.12), but contains $(h_{il}(-t)\,h_{jk}(u-s)+h_{ik}(u-t)\,h_{jl}(-s))$, and similar expressions instead of the analogous expressions with a minus sign.

It remains to find conditions allowing one to prove $(\{c_n\},\alpha)$-mixing.

THEOREM 5. *For a spatially confined quantum system coupled to a quasi-free reservoir (fermion, boson, or classical Gaussian) by an interaction of the form* (3.6), *the* $(\{c_n\},\alpha)$*-mixing condition holds if there are positive constants* κ *and* α *such that*

$$\begin{aligned}&|\langle\phi(T_s v_i)\,\phi(T_t v_j)\rangle|\\&\qquad\leqslant\kappa\exp[-\alpha(t-s)],\qquad i,j=1,\ldots,r,\ t,s\in\mathbb{R};\end{aligned}\tag{3.14}$$

and the coefficients $\{c_n\}$ satisfy the bound

$$c_{2n},\,c_{2n+1}\leqslant c_0(\varepsilon)[8\kappa r^2\,\|X\|^2(1+\varepsilon)\,\alpha^{-1}]^n/n!,\tag{3.15}$$

where $\varepsilon>0$ *and* $\|X\|=\max\{\|X_j\|: j=1,\ldots,r\}$.

Proof. The proof is essentially the same as for Theorem 5 of I, but some additional care is needed because the field operators $\phi(v_j)$ need not commute with each other. Here we shall refer largely to the proof in I, and we shall only discuss the modifications which are needed.

Part (iii) of the mixing condition holds, because of the explicit form (3.11), (3.12) of $K^{(2)}(t)$, $K^{(4)}(t)$, and because of the exponential bound (3.14) on the two-point functions.

In order to prove parts (i) and (ii), the method of I (Theorem 5 and Appendix) can be used, provided one shows that

$$\begin{aligned}&\|A_{01}(t_0)\,A_{11}(t_1)\cdots A_{11}(t_{2n})\,A_{10}(t_{2n+1})\|\\&\qquad\leqslant(4\kappa r^2\,\|X\|^2)^{n+1}\sum_{p\in\mathscr{P}_n'}\prod_{q=0}^{n}\exp[-\alpha(t_{p(2q)}-t_{p(2q+1)})],\end{aligned}\tag{3.16}$$

where $\mathscr{P}'_n$ is the set of those permutations p of $\{0,..., 2n+1\}$ such that $p(2q) < p(2q+1)$, $p(2q) < p(2q+2)$ for all q and such that for each m in $\{1,..., 2n\}$ there is at least one $\bar{q} = \bar{q}(m)$ such that $\bar{p}(2\bar{q}) \leqslant m$, $p(2\bar{q}+1) > m$.

Now we prove (3.16). We recall from [11] that, if $A(t)$ is of the form (3.7), we have, for n even,

$$A(t_0)\, A(t_1) \cdots A(t_{2n})\, A(t_{2n+1})(\rho \otimes |\Omega)(\Omega|)$$
$$= (-1)^{n+1} \sum_{j_0,..., j_{2n+1}=1}^{r} \sum_{[i,k,2n+1]} (-1)^{2n+1-k}$$
$$\times X_{j_{i_0}}(t_{i_0}) \cdots X_{j_{i_k}}(t_{i_k})\, \rho X_{j_{i_{k+1}}}(t_{i_{k+1}}) \cdots X_{j_{i_{2n+1}}}(t_{i_{2n+1}})$$
$$\otimes \phi(T_{t_{i_0}} v_{j_{i_0}}) \cdots \phi(T_{t_{i_k}} v_{j_{i_k}})\, |\Omega)(\Omega|\, \phi(T_{t_{i_{k+1}}} v_{j_{i_{k+1}}}) \cdots \phi(T_{t_{i_{2n+1}}} v_{j_{i_{2n+1}}}),$$

where $\sum_{[i,k,2n+1]}$ extends to the 2^{2n+2} ordered $(2n+2)$-tuples $(i_0,..., i_{2n+1})$ such that $\{i_0,..., i_{2n+1}\}$ coincides with $\{0,..., 2n+1\}$ as a set, and $i_0 < \cdots < i_k$, $i_{k+1} > \cdots > i_{2n+1}$. Then we have (cf. [12])

$$A_{01}(t_0)\, A_{11}(t_1) \cdots A_{11}(t_{2n})\, A_{10}(t_{2n+1})(\rho \otimes |\Omega)(\Omega|)$$
$$= (-1)^{n+1} \sum_{j_0,..., j_{2n+1}=1}^{r} \sum_{[i,k,2n+1]} (-1)^{2n+1-k}$$
$$\times X_{j_{i_0}}(t_{i_0}) \cdots X_{j_{i_k}}(t_{i_k})\, \rho X_{j_{i_{k+1}}}(t_{i_{k+1}}) \cdots X_{j_{i_{2n+1}}}(t_{i_{2n+1}})$$
$$\times \langle \phi(T_{t_{i_{k+1}}} v_{j_{i_{k+1}}}) \cdots \phi(T_{t_{i_{2n+1}}} v_{j_{i_{2n+1}}})\, \phi(T_{t_{i_0}} v_{j_{i_0}}) \cdots \phi(T_{t_{i_k}} v_{j_{i_k}}) \rangle^{\mathrm{QT}}, \tag{3.17}$$

where the "quasi-truncated" correlation function $\langle \cdots \rangle^{\mathrm{QT}}$ is obtained by first expanding the correlation function $\omega_R(\cdots)$ as a sum of products of two-point functions, using formula (3.3) or the similar one without (sgn p), and then deleting the contributions of the terms for which there is an m such that all the time variables $\{t_1,..., t_m\}$ and all the time variables $\{t_{m+1},..., t_{2n+1}\}$ are paired among themselves. For n odd, the left-hand side of (3.17) would vanish because of (3.2).

We parametrize the 4^{n+1} ordered $(2n+2)$-tuples appearing in $\sum_{[i,k,2n+1]}$ by permutations π of $\{0,..., 2n+1\}$ such that $(\pi(0),..., \pi(2n+1)) = (i_{k+1},..., i_{2n+1}, i_0,..., i_k)$. Let $\mathscr{P}'_{n,\pi}$ be the set of those permutations p of $\{0,..., 2n+1\}$ such that $p(2q) < p(2q+1)$, $p(2q) < p(2q+2)$ for all q and such that for each $m = 1,..., 2n$ there is at least a $\bar{q} = \bar{q}(m, \pi)$ such that $\pi p(2\bar{q}) \leqslant m$, $\pi p(2\bar{q}+1) > m$. Then we have, in analogy to (3.3),

$$|\langle \phi(T_{t_{\pi(0)}} v_{j_{\pi(0)}}) \cdots \phi(T_{t_{\pi(2n+1)}} v_{j_{\pi(2n+1)}}) \rangle^{\mathrm{QT}}|$$
$$\leqslant \sum_{p \in \mathscr{P}'_n} \prod_{q=0}^{n} |\langle \phi(T_{t_{\pi p(2q)}} v_{j_{\pi p(2q)}})\, \phi(T_{t_{\pi p(2q+1)}} v_{j_{\pi p(2q+1)}}) \rangle|,$$

and using the bound (3.14) on the two-point functions we find

$$\leqslant \sum_{p \in \mathscr{P}'_{n,\pi}} \prod_{q=0}^{n} \kappa \exp[-\alpha(t_{\pi p(2q)} - t_{\pi p(2q+1)})]$$

$$= [2^{n+1}(n+1)!]^{-1} \sum_{\sigma \in S'_{2n+2}} \prod_{q=0}^{n} \kappa \exp[-\alpha(t_{\sigma(2q)} - t_{\sigma(2q+1)})], \quad (3.18)$$

independently of π, where S'_{2n+2} is the set of those permutations σ of $\{0,..., 2n+1\}$ such that for each $m = 1,..., 2n$ there is at least a $\bar{q} = \bar{q}(m)$ such that $\sigma(2\bar{q}) \leqslant m$, $\sigma(2\bar{q}+1) > m$ (cf. [2, Lemma (I)3.3]).

Now we majorize the norm of (3.17), using (3.18) and the fact that there are 4^{n+1} permutations π to consider. The result is the desired estimate (3.16).

4. Examples

Examples of applications of the method of averaging to the problem of the reduced evolution of an open quantum system can be found in [7, 8]. In [7], a charged harmonic oscillator is coupled to the quantized electromagnetic field in the dipole approximation; in [8], a finite number N of energy levels of an impurity electron is coupled to the phonons of a crystal. Neither model satisfies the $(\{c_n\}, \alpha)$-mixing condition. In [7], the reservoir correlation function is divergent, and renormlization is required, and in [8], exponential decay of the reservoir correlations is forbidden by the fact that the phonons of a crystal have a finite maximum frequency. However, in both cases one might introduce cutoffs and smearings which would make the $(\{c_n\}, \alpha)$-mixing condition hold; for the model of [8], the continuum limit for the crystal should be taken, too.

Here we shall only discuss an extremely simple model, with the purpose of investigating whether the maps $\exp[(\lambda^2 G^{(2)} + \lambda^4 G^{(4)})t]$, $\exp[(\lambda^2 G^{(2)} + \lambda^4 \tilde{G}^{(4)})t]$, $(1 + \lambda^2 M^{(2)}(t)) \exp[(\lambda^2 G^{(2)} + \lambda^4 G^{(4)})t]$ are trace- and positivity-preserving. We know this to be the case for $\exp[\lambda^2 G^{(2)} t]$, which is the same as in the theory of the weak coupling limit, but we have no general argument to ensure a priori the positivity property of the higher-order approximations.

We consider a two-level system S coupled to a boson or fermion reservois R by an interaction Hamiltonian of the form

$$H_I = a^* \otimes a(v) + a \otimes a(v)^*, \quad (4.1)$$

where a^*, a are the creation and annihilation (raising and lowering) operators of the two-level system, satisfying $aa^* + a^*a = 1_S$, and where

$a(v)^*, a(v) = \frac{1}{2}(\phi(v) \pm i\phi(iv))$ are the creation and annihilation operators for a reservoir particle with wavefunction v. We assume that H_S is of the form ωa^*a for some real ω, and that the reservoir is quasi-free (as explained in Section 3), so that we have

$$A(t) = -i[a^* \otimes a(e^{-i\omega t}T_t v) + a \otimes a(e^{-i\omega t}T_t v)^*, \cdot]. \tag{4.2}$$

Let Q be the positive self-adjoint operator on $\mathscr{V}$ such that

$$\langle a(v)^* a(v') \rangle = (v', Qv) \qquad \text{for all } v, v' \text{ in } \mathscr{V}, \tag{4.3}$$

and assume that Q is a function of T_t, so that we have

$$\int_{-\infty}^{+\infty} e^{-i\lambda t}(v, T_t Qv)\, dt = q(\lambda)\, |\hat{v}(\lambda)|^2, \qquad v \in \mathscr{V}, \lambda \in \mathbb{R}, \tag{4.4}$$

where

$$|\hat{v}(\lambda)|^2 = \int_{-\infty}^{+\infty} e^{-i\lambda t}(v, T_t v)\, dt, \qquad v \in \mathscr{V}, \lambda \in \mathbb{R}, \tag{4.5}$$

and where $q(\cdot)$ is a positive function, $0 \leqslant q \leqslant 1$, in the fermion case.

We introduce some notation. For A in $\mathscr{L}(\mathscr{H}_S)$, let

$$L_A(\rho) = A\rho A^* - \tfrac{1}{2}\{A^*A, \rho\}, \qquad \rho \in \mathscr{T}(\mathscr{H}_S), \tag{4.6}$$

and let

$$k(t) = (v, e^{-i\omega t}T_t v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i(\lambda - \omega)t}\, |\hat{v}(\lambda)|^2\, d\lambda, \tag{4.7}$$

$$\begin{aligned} m(t) &= \langle a(v)\, a(e^{-i\omega t}T_t v)^* \rangle \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(\lambda - \omega)t}(1 \pm q(\lambda))\, |\hat{v}(\lambda)|^2\, d\lambda \end{aligned} \tag{4.8}$$

(the minus sign for fermions, the plus sign for bosons),

$$\begin{aligned} n(t) &= \langle a(v)^* a(e^{-i\omega t}T_t v) \rangle \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i(\lambda - \omega)t}\, q(\lambda)\, |\hat{v}(\lambda)|^2\, d\lambda. \end{aligned} \tag{4.9}$$

In order to satisfy the $(\{c_n\}, \alpha)$-mixing condition, we need the functions $|k(t)|$, $|m(t)|$, $|n(t)|$ to be bouned by an exponential $\kappa \exp[-\alpha\, |t|]$; this is essentially a condition on $|\hat{v}(\lambda)|^2$, in particular, the support of $|\hat{v}(\lambda)|^2$ must be the whole real axis.

THEOREM 6. *For the model described above, we have*

$$G^{(2)} = \gamma_-^{(2)}(\omega)\, L_a + \gamma_+^{(2)}(\omega)\, L_{a^*} - i\varepsilon^{(2)}(\omega)[a^*a, \cdot], \tag{4.10}$$

$$G^{(4)} = \gamma_-^{(4)}(\omega)\, L_a + \gamma_+^{(4)}(\omega)\, L_{a^*} + \gamma_0^{(4)}(\omega)\, L_{a^*a} - i\varepsilon^{(4)}(\omega)[a^*a, \cdot], \tag{4.11}$$

$$\tilde{G}^{(4)} = \tilde{\gamma}_-^{(4)}(\omega)\, L_a + \tilde{\gamma}_+^{(4)}(\omega)\, L_{a^*} + \gamma_0^{(4)}(\omega)\, L_{a^*a} - i\varepsilon^{(4)}(\omega)[a^*a, \cdot], \tag{4.12}$$

where

$$\gamma_-^{(2)}(\omega) = (1 \mp q(\omega))\, |\hat{v}(\omega)|^2 \quad (-\text{ for fermions, } + \text{ for bosons}), \tag{4.13}$$

$$\gamma_+^{(2)}(\omega) = q(\omega)\, |\hat{v}(\omega)|^2, \tag{4.14}$$

$$\varepsilon^{(2)}(\omega) = -\frac{1}{2\pi} \mathscr{P} \int_{-\infty}^{+\infty} \frac{\gamma_-^{(2)}(\lambda) + \gamma_+^{(2)}(\lambda)}{\lambda - \omega}\, d\lambda, \tag{4.15}$$

$$\begin{aligned} \tilde{\gamma}_\pm^{(4)}(\omega) = {} & \gamma_+^{(2)}(\omega) \frac{d}{d\omega}\, \varepsilon^{(2)}(\omega) \\ & - \frac{1}{2\pi} \mathscr{P} \int_{-\infty}^{+\infty} \frac{\gamma_\pm^{(2)}(\lambda)}{\lambda - \omega}\, d\lambda\, \frac{d}{d\omega} [\gamma_-^{(2)}(\omega) + \gamma_+^{(2)}(\omega)], \end{aligned} \tag{4.16}$$

$$\begin{aligned} \gamma_\pm^{(4)}(\omega) - \tilde{\gamma}_\pm^{(4)}(\omega) = {} & \pm \Bigg(\gamma_-^{(2)}(\omega)\, \mathscr{P} \int_{-\infty}^{+\infty} \frac{\gamma_+^{(2)}(\lambda) - \gamma_+^{(2)}(\omega)}{(\lambda - \omega)^2}\, d\lambda \\ & - \gamma_+^{(2)}(\omega)\, \mathscr{P} \int_{-\infty}^{+\infty} \frac{\gamma_-^{(2)}(\lambda) - \gamma_-^{(2)}(\omega)}{(\lambda - \omega)^2}\, d\lambda \Bigg), \end{aligned} \tag{4.17}$$

where $\mathscr{P}$ denotes the principal part of the integral, and where, for a fermion reservoir,

$$\gamma_0^{(4)}(\omega) = 0, \qquad \varepsilon^{(4)}(\omega) = -\frac{1}{2} \frac{d}{d\omega} \left[\varepsilon^{(2)}(\omega)^2 - \frac{1}{4} |\hat{v}(\omega)|^4 \right], \tag{4.18}$$

and for a boson reservoir

$$\begin{aligned} \gamma_0^{(4)}(\omega) = -2 \operatorname{Re} J = {} & \frac{1}{\pi} \mathscr{P} \int_{-\infty}^{+\infty} \frac{\gamma_+^{(2)}(\lambda)\, \gamma_-^{(2)}(\lambda) - \gamma_+^{(2)}(\omega)\, \gamma_-^{(2)}(\omega)}{(\lambda - \omega)^2}\, d\lambda \\ & + \gamma_+^{(2)}(\omega) \frac{1}{2\pi} \mathscr{P} \int_{-\infty}^{+\infty} \frac{\gamma_-^{(2)}(\lambda)}{-\lambda - \omega}\, d\lambda \\ & + \gamma_-^{(2)}(\omega) \frac{1}{2\pi} \mathscr{P} \int_{-\infty}^{+\infty} \frac{\gamma_+^{(2)}(\lambda)}{\lambda - \omega}\, d\lambda, \end{aligned} \tag{4.19}$$

$$\varepsilon^{(4)}(\omega) = -\frac{1}{2} \frac{d}{d\omega} \left[\varepsilon^{(2)}(\omega)^2 - \frac{1}{4} (1 + 2q(\omega))^2\, |\hat{v}(\omega)|^4 \right] - \operatorname{Im} J, \tag{4.20}$$

$$J=\int_{t=0}^{\infty}\int_{s=0}^{\infty}[m(-t)\,n(s)+n(t)\,m(-s)]\,s\,ds\,dt$$

$$+\int_{t=0}^{\infty}\int_{s=0}^{t}\int_{u=0}^{s}[m(-t)\,n(u-s)+n(t)\,m(s-u)]\,du\,ds\,dt. \tag{4.21}$$

We have also

$$M^{(2)}(t)\rho=\int_{s=0}^{t}\left\{\int_{u=0}^{s}([a,\rho a^*]\,m(t)\right.$$

$$\left.+\;[a^*,\rho a]\,n(t)+\text{h.c.})\,du-G^{(2)}\rho\right\}ds, \tag{4.22}$$

where ρ is a self-adjoint element of $\mathcal{T}(\mathcal{H}_S)$.

The proof of the Theorem is given in the Appendix.

We see that $G^{(2)}$ has the well-known form, and $G^{(4)}$, $\tilde{G}^{(4)}$ have the same structure, in the case of a fermion reservoir; for a boson reservoir, the fourth-order contribution to the generator is not quadratic in the creation and annihilation operator. All operators $G^{(2)}$, $M^{(2)}(t)$, $G^{(4)}$, $\tilde{G}^{(4)}$ annihilate the trace.

In order to investigate the positivity properties of the semigroups $\{\exp[(\lambda^2G^{(2)}+\lambda^4G^{(4)})t]\colon t\geqslant 0\}$, $\{\exp[(\lambda^2G^{(2)}+\lambda^4\tilde{G}^{(4)})t]\colon t\geqslant 0\}$ we shall need the following Lemma, which is a spacial case of the results of [3, Sect. 4].

LEMMA. *Let $G=\gamma_-L_a+\gamma_+L_{a^*}+\gamma_0L_{a^*a}-i\delta[a^*a,\cdot]$, where γ_-, γ_+, γ_0, ε are real numbers. Then* $\exp[Gt]$, $t>0$, *is completely positive if and only if γ_-, γ_+, $\gamma_0\geqslant 0$, and is positive if and only if γ_-, $\gamma_+\geqslant 0$, $\gamma_0\geqslant -2(\gamma_-\gamma_+)^{1/2}$. If $\gamma_-+\gamma_+>0$, then the semigroup* $\{\exp[Gt]\colon t\geqslant 0\}$ *has a unique stationaty state ρ_0, given by*

$$\rho_0=(\gamma_-+\gamma_+)^{-1}(\gamma_-aa^*+\gamma_+a^*a);$$

if $\gamma_+=\gamma_-=0$, then the stationary states of $\{\exp[Gt]\colon t\geqslant 0\}$ *are all the density matrices commuting with σ_3.*

Then we can prove the following

THEOREM 7. *The semigroups* $\{\exp[(\lambda^2G^{(2)}+\lambda^4G^{(4)})t]\colon t\geqslant 0\}$, $\{\exp[(\lambda^2G^{(2)}+\lambda^4\tilde{G}^{(4)})t]\colon t\geqslant 0\}$ *are completely positive for sufficiently small λ under any of the following conditions*:

(a) *$\gamma_+^{(2)}(\omega)$, $\gamma_-^{(2)}(\omega)$ are both different from* 0,

(b) *either $\gamma_+^{(2)}(\lambda)$ or $\gamma_-^{(2)}(\lambda)$ vanishes identically in λ,*

(c) $\gamma_+^{(2)}(\omega)=\gamma_-^{(2)}(\omega)=0$, *in the case of a fermion reservoir. For a boson reservoir, the semigroups are positive in case* (a) *and completely positive in cases* (b) *and* (c).

The stationary state of the semigroups is unique and faithful in case (a), *unique and pure in case* (b), *and any density matrix commuting with* σ_3 *in case* (c).

Proof. We use the explicit form of $G^{(2)}$, $G^{(4)}$, $\tilde{G}^{(4)}$ and the Lemma. It is clear from (4.13), (4.14) that $\gamma_{\pm}^{(2)}(\omega)$ are non-negative, hence, if they are both non-zero, then $\gamma_{+}^{(2)}(\omega)+\lambda^2\gamma_{\pm}^{(4)}(\omega)$, $\gamma_{+}^{(2)}(\omega)+\lambda^2\tilde{\gamma}_{\pm}^{(4)}(\omega)$ are strictly positive when λ is sufficiently small; if $\lambda^2\gamma_0^{(4)}(\omega)$ does not vanish, still it is small enough to ensure positivity. This proves the statement for case (a). In case (b), we see that $\gamma_{\pm}^{(4)}(\omega)$, $\tilde{\gamma}_{\pm}^{(4)}(\omega)$ vanish if $\gamma_{\pm}^{(2)}(\lambda)$ is identically zero (upper or lower signs must be taken together), and also $\gamma_0^{(4)}(\omega)$ vanishes. In case (c), also $\gamma_{\pm}^{(4)}(\omega)$, $\gamma_{\pm}^{(4)}(\omega)$ vanish, and one can see from (4.19) that $\gamma_0^{(4)}(\omega)$ is non-negative. The result follows, using the Lemma.

Remarks. Condition (b) holds when the reference state of the reservoir is the vacuum, with $q(\lambda)=0$ for all λ, or also, in the fermion case, when $q(\lambda)=1$ for all λ. When the reference state of the reservoir is KMS at some inverse temperature $\beta\in\mathbb{R}$, then either condition (a) or condition (c) holds, depending on whether $|\hat{v}(\omega)|^2>0$ or $|\hat{v}(\omega)|^2=0$. In case (a), it need not be the case that the stationary state of $\exp[(\lambda^2G^{(2)}+\lambda^4G^{(4)})t]$, or of $\exp[(\lambda^2G^{(2)}+\lambda^4\tilde{G}^{(4)})t]$, is the canonical state at inverse temperature β for the free dynamics of the system; in general, it depends on the coupling constant λ and on the form of the function $|\hat{v}(\cdot)|^2$, and it approaches the canonical state $\exp[-\beta\omega a^*a]/\mathrm{tr}\{\exp[-\beta\omega a^*a]\}$ in the limit as $\lambda\to 0$.

In the case of a boson reservoir, it is possible to have $\gamma_0^{(4)}(\omega)=0$ under condition (a), then the fourth-order semigroup is positive but not completely positive.

We have no general method to prove positivity of the maps $(1+\lambda^2M^{(2)}(t))\exp[(\lambda^2G^{(2)}+\lambda^4G^{(4)})t]$. In the special case of a fermion reservoir with a constant $q(\lambda)$, we are able to give a proof through a different method, which we shall now describe.

When the reservoir is made of fermions, one might assume that the creation and anihilation operators of the reservoir anticommute with a and a^*; this can be obtained by representing $a(v)$, $a(v)^*$ on $\mathscr{H}_S\otimes\mathscr{H}_R$ as $\sigma_3\otimes a(v)$, $\sigma_3\otimes a(v)^*$ and letting

$$\begin{aligned} H_I &= a^*a(v)+a(v)^*a = a^*\sigma_3\otimes a(v)+\sigma_3 a\otimes a(v)^* \\ &= -(a^*\otimes a(v)+a\otimes a(v)^*). \end{aligned} \tag{4.23}$$

The sign of H_I is irrelevant as far as the reduced dynamics of S is concerned, and the results of the previous discussion remain unchanged. But now an alternative approach becomes possible: the composite system $S+R$ is a quasi-free fermion system, and its dynamics is the second quantization of a group of unitaries on a Hilbert space. If we denote by e a unit vector, orthogonal to $\mathscr{K}$, and by D the infinitesimal generator of $\{T_t : t \in \mathbb{R}\}$, this group of unitaries is $\{\exp[i(\omega\,|e\rangle\langle e| + \lambda(|e\rangle\langle v| + |v\rangle\langle e|) + D)t] : t \in \mathbb{R}\}$ on $\mathbb{C}e \oplus \mathscr{K}$. In the interaction picture, we obtain the equation

$$\frac{d}{dt} f^{\lambda}(t) = \lambda A(t)\, f^{\lambda}(t)$$

in $\mathbb{C}e \oplus \mathscr{K}$, where

$$A(t) = i(e^{-i\omega t}\,|e\rangle\langle T_{-t}v| + e^{i\omega t}\,|T_{-t}v\rangle\langle e|). \tag{4.24}$$

The application of the averaging method to this Hilbert space problem is extremely simple, since $A_{11}(t)$ vanishes identically. Strictly speaking, the $(\{c_n\}, \alpha)$-mixing condition is not really necessary, and it suffices that $\int_0^\infty |k(t)|\, t^r\, dt$ exists for all positive r. Straightforward computations, similar to those in the Appendix, lead to the following

THEOREM 8. *There is a positive function $\beta_4(\cdot)$, bounded on compacts, such that*

$$\|P_0 f^{\lambda}(t) - (1 + \lambda^2 m^{(2)}(t)) \exp[(\lambda^2 g^{(2)} + \lambda^4 g^{(4)})t]e\| \leqslant \beta_4(\lambda^2 t), \tag{4.25}$$

where $f_0 = e$, $P_0 = |e\rangle\langle e|$, and where

$$g^{(2)} = -\int_0^\infty k(t)\, dt = -\frac{1}{2}\,|\hat{v}(\omega)|^2 + i\varepsilon^{(2)}(\omega),$$

$$m^{(2)}(t) = \int_0^t k(s)s\, ds + \int_t^\infty (t-s)\, k(s)\, ds$$

$$\xrightarrow[t \to \infty]{} i\frac{d}{d\omega}\left[\frac{1}{2}\,|\hat{v}(\omega)|^2 - i\varepsilon^{(2)}(\omega)\right],$$

$$g^{(4)} = g^{(2)} m^{(2)}(\infty) = -i\,\frac{i}{2}\frac{d}{d\omega}\left(\frac{1}{2}\,|\hat{v}(\omega)|^2 - i\varepsilon^{(2)}(\omega)\right)^2.$$

Assume $|\hat{v}(\omega)|^2 > 0$. Then $|\exp[(\lambda^2 g^{(2)} + \lambda^4 g^{(4)})t]| < 1$ for all $t > 0$, at least for λ sufficiently small; as a consequence, also $|(1 + \lambda^2 m^{(2)}(t)) \exp[(\lambda^2 g^{(2)} + \lambda^4 g^{(4)})t]| \leqslant 1$, at least for λ sufficiently small and t sufficiently large. Then, for each q in $[0, 1]$, there exists a completely positive quasi-free map [13] on $\mathscr{L}(\mathscr{H}_S)$, denoted by $Z^{\lambda}_{q,t}$, such that

$$Z^{\lambda}_{q,t}(a^*) = (1 + \lambda^2 m^{(2)}(t)) \exp[(\lambda^2 g^{(2)} + \lambda^4 g^{(4)})t]\, a^*$$

and

$$\lim_{t \to \infty} Z^{\lambda}_{q,t}(a^*a) = q1.$$

By inspection, one sees that, if $q(\lambda) = q$ *for all* λ *in* $\mathbb{R}$, *then* $Z^{\lambda}_{q,t}$ *is the dual map of* $(1 + \lambda^2 M^{(2)}(t)) \exp[(\lambda^2 G^{(2)} + \lambda^4 G^{(4)})t]$, *defined by* (4.10), (4.11), *and* (4.22); *this proves complete positivity of the latter map. No such result holds when* $q(\lambda)$ *is not constant.*

This situation should be contrasted with the weak coupling limit theory for quasi-free systems (see, e.g., [14]). It is completely equivalent to perform the weak coupling limit on the evolution equation for density matrices or on the underlying equation on the test function space, irrespectively of whether $q(\lambda)$ is constant or not, and $q(\lambda)$ affects the reduced dynamics in the weak coupling limit only through its value at $\lambda = \omega$. When higher-order corrections are considered, the methods of [14] allow one to obtain norm estimates on $P_0 f^{\lambda}(t) - y_4^{\lambda}(t)$ from estimates of the form (4.25) on the test function space only when $q(\lambda)$ is constant, and the form of $q(\lambda)$ becomes important, as is apparent from (4.16), (4.17). This is connected to the phenomenon that might be called "quantum thermal memory": if $|\hat{v}(\omega)|^2$ depends on an additional parameter ε and tends to a constant when ε goes to zero, then the semigroup approximation to $P_0 f^{\lambda}(t)$ becomes exact if the reservoir is in the vacuum state $(q = 0)$, but it does not when the reservoir is in a KMS state at some inverse temperature $\beta \in \mathbb{R}$. Accordingly, when $|\hat{v}(\omega)|^2$ tends to a constant, $m^{(2)}(t)$ vanishes, but $M^{(2)}(t)$ does not, when $q(\lambda)$ is not constant.

APPENDIX: PROOF OF THEOREM 6

We have

$$K^{(2)}(t)\rho = [\rho a^*, a]\, m(t) + [\rho a, a^*]\, n(t) + \text{h.c.},$$

so that $K^{(2)}(t)$ commutes with the free evolution. Then we get

$$G^{(2)}\rho = \int_0^{\infty} \{[a, \rho a^*]\, m(t) + [a^*, \rho a]\, n(t) + \text{h.c.}\}\, dt,$$

$$M^{(2)}(\infty)\rho = \int_0^{\infty} \{[\rho a^*, a]\, m(s) + [\rho a, a^*]\, n(s) + \text{h.c.}\} s\, ds.$$

We compute $G^{(2)}M^{(2)}(\infty)$, $M^{(2)}(\infty)\,G^{(2)}$, to find

$$\begin{aligned} G^{(2)}M^{(2)}(\infty)\rho = \int_{t=0}^{\infty}\int_{s=0}^{\infty} & \{[a, \rho a^*]\, m(t)\, m(s) + \text{h.c.} \\ & + (a\rho a^* - a^*a\rho a^*a)[2\,\mathrm{Re}[m(t)\, m(-s)] \\ & + 2\,\mathrm{Re}\, n(t)\, 2\,\mathrm{Re}\, m(s)] \\ & - aa^*\rho a^*a[m(t)\, n(-s) + n(-t)\, m(s)]\}\, s\, ds\, dt \\ & + (+ \rightleftharpoons -), \end{aligned}$$

where $(+ \rightleftharpoons -)$ denotes a similar expression, with $a, a^*; m, n$ interchanged,

$$\begin{aligned} M^{(2)}(\infty)\, G^{(2)}\rho = \int_{t=0}^{\infty}\int_{s=0}^{\infty} & \{[a, \rho a^*]\, m(t)\, m(s) + \text{h.c.} \\ & + (a\rho a^* - a^*a\rho a^*a)[2\,\mathrm{Re}[m(-t)\, m(s)] \\ & + 2\,\mathrm{Re}\, m(t)\, 2\,\mathrm{Re}\, n(s)] \\ & - aa^*\rho a^*a[n(-t)\, m(s) + m(t)\, n(-s)]\}\, s\, ds\, dt \\ & + (+ \rightleftharpoons -), \end{aligned}$$

so that

$$\begin{aligned} [G^{(2)}, M^{(2)}(\infty)]\rho = \int_{t=0}^{\infty}\int_{s=0}^{\infty} & (2\,\mathrm{Re}\, n(t)\, 2\,\mathrm{Re}\, m(s) \\ & - 2\,\mathrm{Re}\, m(t)\, 2\,\mathrm{Re}\, n(s))s\, ds\, dt \\ & \times (a\rho a^* - a^*a\rho a^*a - a^*\rho a + aa^*\rho aa^*). \end{aligned}$$

Now we add and subtract $-\frac{1}{2}\{a^*a, \rho\} + \frac{1}{2}\{aa^*, \rho\}$, to obtain

$$a\rho a^* - a^*a\rho a^*a - a^*\rho a - aa^*\rho aa^* = L_a\rho - L_{a^*a}\rho - L_{a^*}\rho + L_{aa^*}\rho,$$

and use $L_{aa^*} = L_{a^*a}$ to find

$$\begin{aligned} [G^{(2)}, M^{(2)}(\infty)] = \int_{t=0}^{\infty}\int_{s=0}^{\infty} & (2\,\mathrm{Re}\, n(t)\, 2\,\mathrm{Re}\, m(s) \\ & - 2\,\mathrm{Re}\, m(t)\, 2\,\mathrm{Re}\, n(s))s\, ds\, dt(L_a - L_{a^*}). \end{aligned}$$

Also $K^{(4)}(t)$ commutes with the free evolution, and we must compute $\int_0^\infty K^{(4)}(t)\, dt$. For a fermion reservoir, the result is

$$\int_0^\infty K^{(4)}(t)\,\rho\,dt=\int_{t=0}^{\infty}\int_{s=0}^{t}\int_{u=0}^{s}\{[a,a^*\rho]\,n(-t)\,m(u-s)+\text{h.c.}$$

$$-\ a^*a\rho aa^*(n(t)\,m(s-u)+n(u-s)\,m(-t))$$

$$+\ (a^*a\rho a^*a-a\rho a^*)\,2\,\mathrm{Re}[m(t)\,m(u-s)+m(t-u)\,m(-s)$$

$$+\ n(u-t)\,m(-s)]+(+\rightleftharpoons-)\}\,du\,ds\,dt; \tag{A.1}$$

whereas for a boson reservoir there is an additional term, given by

$$2a^*a\rho aa^*\int_{t=0}^{\infty}\int_{s=0}^{t}\int_{u=0}^{s}\{n(t)\,m(u-s)+n(s)\,m(u-t)$$

$$+m(-t)\,n(s-u)+m(-s)\,n(t-u)$$

$$+n(t)\,m(s-u)+m(-t)\,n(u-s)\}\,du\,ds\,dt+(+\rightleftharpoons-).$$

Due to the presence of the operation $(+\rightleftharpoons-)$, we may replace $-a^*a\rho aa^*[n(t)\,m(s-u)+n(u-s)\,m(-t)]$ with $-a^*a\rho aa^*n(t)\,m(s-u)-aa^*\rho a^*an(-t)\,m(u-s)$ in (A.1); using the anticommutation relation $aa^*+a^*a=1$ we find

$$(\text{A.1})=(a^*a\rho a^*a-a\rho a^*)\,2\,\mathrm{Re}\int_{t=0}^{\infty}\int_{s=0}^{t}\int_{u=0}^{s}\{m(t)\,m(u-s)$$

$$+m(t-u)\,m(-s)+n(u-t)\,m(-s)+n(-t)\,m(u-s)\}\,du\,ds\,dt$$

$$=(a^*a\rho a^*a-a\rho a^*)$$

$$\times 2\,\mathrm{Re}\int_{t=0}^{\infty}\int_{s=0}^{\infty}\{m(t)\,m(-s)+n(-t)\,m(-s)\}t\,dt\,ds,$$

where we have used the identity

$$\int_{t=0}^{\infty}\int_{s=0}^{t}\int_{u=0}^{s}\{f(t)\,g(s-u)+f(t-u)\,g(s)\}\,du\,ds\,dt$$

$$=\int_0^\infty f(t)t\,dt\int_0^\infty g(s)\,ds, \tag{A.2}$$

which can be proved in the same way as the corresponding identity (4.15) of I.

The additional term for bosons can be worked out by taking into account the fact that the interchange $(+\rightleftharpoons-)$ is the same as hermitian conjugation for it, and by using $a^*a\rho aa^* = a^*a\rho - a^*a\rho a^*a$ and the identity (A.2) once again. The result is

$$-2\,\mathrm{Re}\,JL_{a^*a}\rho + i\,\mathrm{Im}\,J[a^*a, \rho], \tag{A.3}$$

where J has been defined in (4.21).

So we find, for a fermion reservoir,

$$\begin{aligned}
\tilde{G}^{(4)}\rho = \int_{t=0}^{\infty}\int_{s=0}^{\infty} &\{[a, \rho a^*]\, m(t)\, m(s) + \text{h.c.} \\
&- a^*a\rho aa^*(m(-t)\, n(s) + n(t)\, m(-s)) + (a\rho a^* - a^*a\rho a^*a) \\
&\times [m(t)\, n(-s) + m(-t)\, n(s)] + (+ \rightleftarrows -)\}s\, ds\, dt \\
= \int_{t=0}^{\infty}\int_{s=0}^{\infty} &\{a\rho a^*\, 2\,\mathrm{Re}[m(t)(m(s) + n(-s))] \\
&+ a^*\rho a[n(t)(n(s) + m(-s))] - a^*a\rho m(-t)(m(-s) + n(s)) \\
&- aa^*\rho n(-t)(n(-s) + m(s)) - \rho a^*am(t)(m(s) + n(-s)) \\
&- \rho aa^*n(t)(n(s) + m(-s))\}s\, ds\, dt,
\end{aligned}$$

and for a boson reservoir we must add the term (A.3).

Then the announced results (4.10)–(4.22) are found by using the following calculations:

$$\begin{aligned}
\int_0^{\infty} m(t)\, dt &= \int_{-\infty}^{+\infty} \left[\frac{1}{2\pi}\int_{t=0}^{\infty} e^{i(\lambda-\omega)t}\, dt\right](1 \mp q(\lambda))\, |\hat{v}(\lambda)|^2\, d\lambda \\
&= \lim_{\varepsilon\downarrow 0} \frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda - \omega + i\varepsilon}(1 \mp q(\lambda))\, |\hat{v}(\lambda)|^2\, d\lambda \\
&= \frac{1}{2}(1 \mp q(\omega))\, |\hat{v}(\omega)|^2 + \frac{i}{2\pi}\mathscr{P}\int_{-\infty}^{+\infty} \frac{(1 \mp q(\lambda))\, |\hat{v}(\lambda)|^2}{\lambda - \omega}\, d\lambda \\
&\equiv \frac{1}{2}\gamma^{(2)}(\omega) + \frac{i}{2\pi}\mathscr{P}\int_{-\infty}^{+\infty} \frac{\gamma^{(2)}(\lambda)}{\lambda - \omega}\, d\lambda;
\end{aligned}$$

$$\begin{aligned}
\int_0^{\infty} n(t)\, dt &= \int_{-\infty}^{+\infty} \left[\frac{1}{2\pi}\int_{t=0}^{\infty} e^{-i(\lambda-\omega)t}\, dt\right] q(\lambda)\, |\hat{v}(\lambda)|^2\, d\lambda \\
&= -\lim_{\varepsilon\downarrow 0} \frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda - \omega - i\varepsilon}\, q(\lambda)\, |\hat{v}(\lambda)|^2\, d\lambda \\
&= \frac{1}{2} q(\omega)\, |\hat{v}(\omega)|^2 - \frac{i}{2\pi}\mathscr{P}\int_{-\infty}^{+\infty} \frac{q(\lambda)\, |\hat{v}(\lambda)|^2}{\lambda - \omega}\, d\lambda \\
&\equiv \frac{1}{2}\gamma_+^{(2)}(\omega) - \frac{i}{2\pi}\mathscr{P}\int_{-\infty}^{+\infty} \frac{\gamma_+^{(2)}(\lambda)}{\lambda - \omega}\, d\lambda;
\end{aligned}$$

$$\int_0^{\infty} m(s)s\,ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{s=0}^{\infty} e^{i(\lambda-\omega)s}s\,ds\right](1\mp q(\lambda))\,|\hat{v}(\lambda)|^2\,d\lambda$$

$$= i\frac{d}{d\omega}\int_0^{\infty} m(t)\,dt$$

$$= -\frac{1}{2\pi}\frac{d}{d\omega}\mathscr{P}\int_{-\infty}^{+\infty}\frac{\gamma_-^{(2)}(\lambda)}{\lambda-\omega}\,d\lambda + \frac{i}{2}\frac{d}{d\omega}\gamma_-^{(2)}(\omega)$$

$$= -\frac{1}{2\pi}\mathscr{P}\int_{-\infty}^{+\infty}\frac{\gamma_-^{(2)}(\lambda)-\gamma_+^{(2)}(\omega)}{(\lambda-\omega)^2}\,d\lambda + \frac{i}{2}\frac{d}{d\omega}\gamma_-^{(2)}(\omega);$$

$$\int_0^{+\infty} n(s)s\,ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{s=0}^{\infty} e^{-i(\lambda-\omega)s}s\,ds\right]q(\lambda)\,|\hat{v}(\lambda)|^2\,d\lambda$$

$$= -i\frac{d}{d\omega}\int_0^{\infty} n(t)\,dt$$

$$= -\frac{1}{2\pi}\frac{d}{d\omega}\mathscr{P}\int_{-\infty}^{+\infty}\frac{\gamma_+^{(2)}(\lambda)}{\lambda-\omega}\,d\lambda - \frac{i}{2}\frac{d}{d\omega}\gamma_+^{(2)}(\omega)$$

$$= -\frac{1}{2\pi}\mathscr{P}\int_{-\infty}^{+\infty}\frac{\gamma_+^{(2)}(\lambda)-\gamma_+^{(2)}(\omega)}{(\lambda-\omega)^2}\,d\lambda - \frac{i}{2}\frac{d}{d\omega}\gamma_+^{(2)}(\omega).$$

$\gamma_-^{(2)}(\lambda)$, $\gamma_+^{(2)}(\lambda)$ are non-negative for all λ, and if they vanish at ω, then also their first derivatives vanish at ω. To compute $\operatorname{Re} J$, we use the notation

$$\hat{f}(\lambda) = \int_{-\infty}^{+\infty} e^{-i\lambda t} f(t)\,dt, \qquad (\mathscr{H}g)(\lambda) = \frac{1}{2\pi}\mathscr{P}\int_{-\infty}^{+\infty}\frac{g(\mu)}{\mu-\lambda}\,d\mu$$

and we find

$$2\operatorname{Re} J = 2\operatorname{Re}\left(\int_0^{\infty} m(-t)\,dt\int_0^{\infty} n(s)s\,ds\right.$$

$$\left.+\int_{t=0}^{\infty}\int_{s=0}^{t}\int_{u=0}^{s} m(-t)\,n(u-s)\,du\,ds\,dt\right) + (m \rightleftharpoons n)$$

$$= -\hat{m}(0)(\mathscr{H}\hat{n})(0) + (\mathscr{H}\hat{m}(0))\,\hat{n}(0)$$

$$+\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\hat{m}(\lambda)\,\hat{n}(u)$$

$$\times\left[\iiint_{0\leqslant u\leqslant s\leqslant t<\infty} e^{-i\lambda t-i\mu s+i\mu u}\,du\,ds\,dt\right]d\lambda\,d\mu + (m \rightleftharpoons n)$$

and with the usual kind of manipulations, this becomes

$$\frac{1}{\pi}\mathscr{P}\int_{-\infty}^{+\infty} \lambda^{-2}[\hat{m}(0)\,\hat{n}(0)-\hat{m}(\lambda)\,\hat{n}(-\lambda)]\,d\lambda+(\mathscr{H}\hat{m})(0)\,\hat{n}(0)+(\mathscr{H}\hat{u})(0)\,\hat{m}(0)$$

$$=\frac{1}{\pi}\mathscr{P}\int_{-\infty}^{+\infty}\frac{\gamma_{+}^{(2)}(\omega)\,\gamma^{(2)}(\omega)-\gamma_{+}^{(2)}(\lambda)\,\gamma^{(2)}(\lambda)}{(\lambda-\omega)^{2}}\,d\lambda$$

$$+\gamma_{+}^{(2)}(\omega)\,\frac{1}{2\pi}\mathscr{P}\int_{-\infty}^{+\infty}\frac{\gamma^{(2)}(\lambda)}{\lambda-\omega}\,d\lambda+\gamma^{(2)}(\omega)\,\frac{1}{2\pi}\mathscr{P}\int_{-\infty}^{+\infty}\frac{\gamma_{+}^{(2)}(\lambda)}{\lambda-\omega}\,d\lambda.$$

REFERENCES

1. A. FRIGERIO, J. T. LEWIS, AND J. V. PULÈ, The averaging method for asymptotic evolutions. I. Stochastic differential equations, *Adv. in Appl. Math.* **2** (1981), 456–481.
2. E. B. DAVIES, Markovian master equations, *Comm. Math. Phys.* **39** (1974), 91–110; II, *Math. Ann.* **219** (1976), 147–158; III, *Ann. Inst. H. Poincaré Sect. B* **11** (1975), 265–273.
3. V. GORINI, A. FRIGERIO, M. VERRI, A. KOSSAKOWSKI, AND E. C. G. SUDARSHAN, Properties of quantum Markovian master equations, *Rep. Math. Phys.* **13** (1978), 149–173.
4. H. SPOHN AND J. L. LEBOWITZ, Irreversible thermodynamics for quantum systems weakly coupled to thermal reservoirs, *Adv. in Chem. Phys.* **38** (1978), 109–142.
5. H. SPOHN, Kinetic equations from Hamiltonian dynamics: Marovian limits, *Rev. Modern Phys.* **53** (1980), 569–615.
6. G. W. FORD, J. T. LEWIS, AND J. R. MCCONNELL, Rotational Brownian motion of an asymmetric top, *Phys. Rev. A* **19** (1979), 907–919.
7. G. W. FORD AND J. T. LEWIS, A Bethe-type calculation of the Lamb shift for the harmonic oscillator, in preparation.
8. A. BARCHIELLI, E. MULAZZI, AND G. PARRAVICINI "Open System Approach to Jahn–Teller Systems," Preprint I.F.U.M. 251/F.T., University of Milan, 1980.
9. T. KATO, "Perturbation Theory for Linear Operators," Springer-Verlag, Berlin/Heidelberg/New York, 1966.
10. E. B. DAVIS AND J. P. ECKMANN, Time decay for fermion systems with persistent vacuum, *Helv. Phys. Acta* **48** (1975), 731–742.
11. V. GORINI AND A. KOSSAKOWSKI, *N*-Level system in contact with a singular reservoir, *J. Math. Phys.* **17** (1976), 1298–1305.
12. A. FRIGERIO, C. NOVELLONE, AND M. BERRI, Master equation treatment of the singular reservoir limit, *Rep. Math. Phys.* **12** (1977), 275–284.
13. D. E. EVANS, Completely positive quasi-free maps on the CAR algebra, *Comm. Math. Phys.* **70** (1979), 53–60.
14. A. FRIGERIO, V. GORINI, AND J. V. PULÈ, Open quasi-free systems, *J. Statist. Phys.* **22** (1980), 409–433.